

Appendix: Proof of Spatial Derivative of Clear Raindrop

Shaodi You[†] Robby T. Tan[‡]

Rei Kawakami[†] Katsushi Ikeuchi[†]

[†]The University of Tokyo

[‡]Utrecht University

{yousd, rei, ki}@cvl.iis.u-tokyo.ac.jp

R.T.Tan@uu.nl

Layout Section 1 forms the imagery of clear raindrop with more detailed notations. Two conclusions used in the manuscript (Eq. (4), after Eq. (6)) are at the end of Section 2 and Section 3 correspondingly.

1. Geometric Derivative of Clear Raindrop

Continuous mapping As shown in Fig. 1 (a), the appearance of each raindrop is a contracted image of the background, as if it is taken from a catadioptric camera. The numeric values indicated in Fig. 1 (c) are the contraction ratios between the original image and the image inside the raindrops, calculated from the black and white patterns. The contraction ratio is around 20 to 30, meaning that the motion observed inside the raindrops will be 1/30 to 1/20 smaller than the other areas in the image. This significant motion difference can be used as a clue for raindrop detection.

Mathematically, for a given raindrop, we describe the smooth expand mapping start from raindrop area Ω_r into the environment scene Ω_e as φ :

$$\varphi : \Omega_r \rightarrow \Omega_e. \quad (12)$$

The appearance of the raindrop and the environment share the same image plane and coordinates. In order to distinguish, we denote the points and coordinates in raindrop Ω_r as: $\mathbf{P}_r = (u, v)$ and the corresponding points and coordinates in environment Ω_e as $\mathbf{P}_e = (x, y)$. Then φ can be expressed as:

$$\mathbf{P}_e = (x, y) = \varphi(\mathbf{P}_r) = \varphi(u, v) = (\varphi^1(u, v), \varphi^2(u, v)). \quad (13)$$

Local derivative (Jacobian) The local differentials (Jacobian) of φ at $\mathbf{P}_r = (u, v)$ is defined as:

$$J_\varphi(\mathbf{P}_r) = J_\varphi(u, v) = \begin{pmatrix} \varphi_u^1(u, v) & \varphi_v^1(u, v) \\ \varphi_u^2(u, v) & \varphi_v^2(u, v) \end{pmatrix}, \quad (14)$$

with: $\varphi_u^1(u, v) = \frac{\partial \varphi^1(u, v)}{\partial u}$.

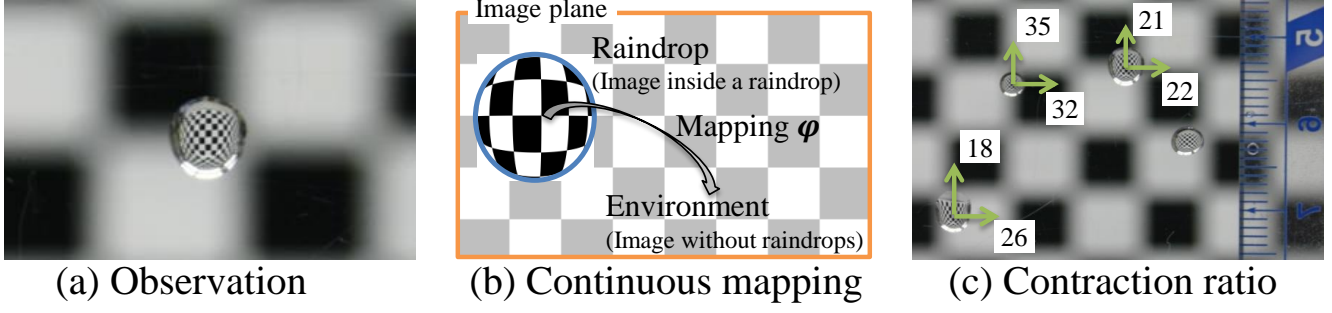


Figure 1. (a) The appearance of each raindrop is a contracted image of the background. (b) On the image plane, there is a smooth mapping φ starting from the raindrop into the environment scene. (c) Contraction ratios from background to raindrop are significant.

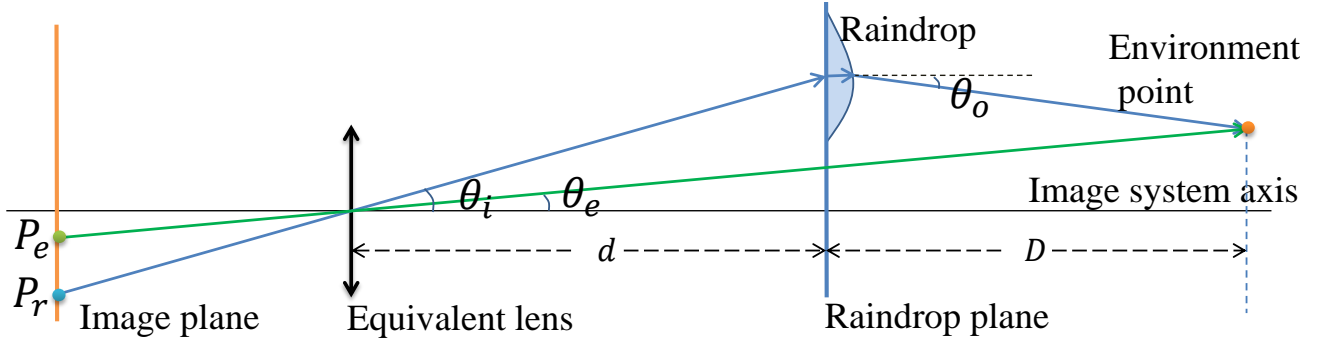


Figure 2. Refraction model of a pair of corresponding points on an image plane. There are two refractions on the light path through a raindrop. (The camera lens cover or protecting shield is assumed to be a thin plane and thus neglected.)

The local motion at (u, v) , denoted as $(\delta u, \delta v)^T$, and the local motion at (x, y) , denoted as $(\delta x, \delta y)^T$, is linearly associated by $J_\varphi(u, v)$:

$$\begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = J_\varphi(u, v) \begin{pmatrix} \delta u \\ \delta v \end{pmatrix}. \quad (15)$$

Instead of modeling φ or $J_\varphi(u, v)$, we are interested in the non-directional scale ratio between $\|(\delta x, \delta y)\|$ and $\|(\delta u, \delta v)\|$. According to Eq.(15):

$$\|(\delta x, \delta y)\|^2 = (\delta x, \delta y)^T (\delta x, \delta y) = (\delta u, \delta v)^T (J_\varphi(u, v))^T J_\varphi(u, v) (\delta u, \delta v), \quad (16)$$

with $(J_\varphi(u, v))^T J_\varphi(u, v)$ is symmetric and positive-semidefinite, and can be diagonalized as:

$$(J_\varphi(u, v))^T J_\varphi(u, v) = E^T \begin{pmatrix} \lambda_1^2(u, v) & \\ & \lambda_2^2(u, v) \end{pmatrix} E, \quad (17)$$

where E is an orthogonal matrix, and $0 \leq \lambda_1(u, v) < \lambda_2(u, v)$. Therefore, according to Eqs.(16) and (17), for any directional motion $(\delta u, \delta v)$ at (u, v) :

$$\frac{\|(\delta x, \delta y)\|}{\|(\delta u, \delta v)\|} \geq \lambda_1(u, v). \quad (18)$$

A simplified version of Eq. (18) in the manuscript (Eq.(3)) is given as:

$$\mathcal{E}_\varphi(u, v, \delta u, \delta v) = \lim_{(\delta u, \delta v) \rightarrow 0} \left| \frac{\varphi(u + \delta u, v + \delta v) - \varphi(u, v)}{(u + \delta u, v + \delta v) - (u, v)} \right|.$$

We can give a lower boundary, denoted as λ_{lower} , for all $\lambda_1(u, v)$ inside the raindrop area Ω_r :

$$\lambda_{lower} \leq \min\{\lambda_1(u, v) | (u, v) \in \Omega_r\} \quad (19)$$

We call it the lower boundary of the expansion ratio.

2. Proof of Expansion Ratio

Lower boundary of the expansion ratio The lower boundary of the expansion ratio is given by:

$$\lambda_{lower} \geq \frac{n_a}{n_w} \left| 1 - \frac{n_w}{n_a} \frac{d}{R} \frac{D}{d + D} \right|, \quad (20)$$

where n_w and n_a are refraction indices of water and air, d is the distance from the lens to the raindrop, D is the distance from the raindrop to the background, and R is the lower boundary of raindrop curvature radius.

Proof A light ray passing through a raindrop undergoes two refractions: first, the refraction from the air to the water, and second, the refraction from the water to the air. Thus, the mapping function, φ , can be separated as two continuous mappings:

$$\varphi = \varphi^{a \rightarrow w} \circ \varphi^{w \rightarrow a}, \quad (21)$$

index a stands for air, and w stands for water.

Assuming the contact surface between the camera lens cover and raindrop is flat, $\varphi^{a \rightarrow w}$ should be analytically solvable.

According to Snell's law, where $n_a \sin \theta_i = n_w \sin \theta_o$, we can have:

$$\frac{P_e}{P_r} = \frac{d \tan \theta_e}{d \tan \theta_i} = \frac{n_a}{n_w} \frac{1 + \frac{n_w}{n_a} \frac{d}{D}}{1 + \frac{d}{D}} = \text{constant}, \quad (22)$$

where the notation is defined in Fig. 2, and $\frac{n_w}{n_a}$ is approximately $\frac{4}{3} > 1$. Thus the ratio:

$$\frac{dP_e}{dP_r} = \frac{P_e}{P_r} = \frac{n_a}{n_w} \frac{1 + \frac{n_w}{n_a} \frac{d}{D}}{1 + \frac{d}{D}} > \frac{n_a}{n_w}. \quad (23)$$

Hence, a lower boundary of the expansion (contraction here) ratio is:

$$\lambda_{lower}^{a \rightarrow w} = \frac{n_a}{n_w}. \quad (24)$$

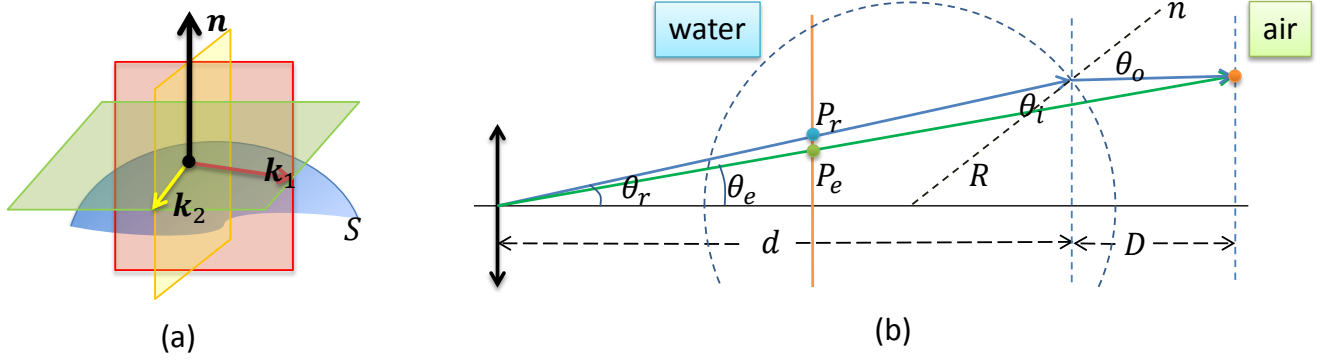


Figure 3. Simplified refraction model of the second refraction using principle curvature. (a) Given a point on raindrop surface S , its two principle curvatures vectors k_1 , k_2 and the normal n are orthogonal to each other. (b) Refraction model of the second refraction when assuming the normal of refraction is very close to the image system axis. The notation are same with Fig. 2, R is the curvature radius at the place and direction where the refraction happens.

Now, we estimate the expansion ratio of the second refraction φ^{w-a} . Note that, referring to Fig. 2, although in the first refraction the direction and position of the emergence light trace could be analytically solved, the position and angle of the incident light of the second refraction is still unsolvable. This is because we have no knowledge of the position and shape of the raindrop.

To estimate the expansion ratio of the second refraction, we start from the differential geometry on the outer surface of the raindrop. For a given position (u, v) on the surface of the raindrop, its up to second order differential geometry values are illustrated as in Fig. 3(a) [2]. The upper principle curvature vector, k_1 , points to the direction where the raindrop surface bends most. And the lower principle curvature vector, k_2 , points to the direction where the surface bends least. The curvature vector of any other direction, k , is the linear combination of k_1 and k_2 . The values of any curvature vector k is bounded by k_1 and k_2 :

$$k_2 \leq k \leq k_1. \quad (25)$$

The reciprocal of curvature is called curvature radius: $R = \frac{1}{k}$. In any direction, it is bounded by two principle curvature radius: $R_1 \leq R \leq R_2$.

As illustrated in Fig. 2, we now consider the second refraction locally at given point (u, v) . Mention that there is no knowledge about how this local coordinates is aligned to the global coordination. First, we try to estimate the angular ratio $\frac{d\theta_o}{d\theta_i}$.

According to Snell's law, we have:

$$\frac{d\theta_o}{d\theta_i} = \frac{\frac{n_w}{n_a} \cos \theta_i}{\left(1 - \left(\frac{n_w}{n_a}\right)^2 \sin^2 \theta_i\right)^{\frac{1}{2}}}, \quad (26)$$

where we know that $\frac{n_w}{n_a} > 1$, thus Eq.(26) gets its minimum when $\theta_i=0$:

$$\min\left(\frac{d\theta_o}{d\theta_i}\right) = \left.\frac{d\theta_o}{d\theta_i}\right|_{\theta_i=0}. \quad (27)$$

This is in accordance with the observation of real raindrop image shown in Fig. 1(a).

As illustrated in Fig. 3(b), according to Eq.(27), we may put the normal of the raindrop surface considerably close to the image system axis. Assuming every angle is significantly small:

$$\theta_i \ll 1, \theta_o \ll 1, \theta_e \ll 1, \theta_r \ll 1. \quad (28)$$

According to Eq.(28), we can use the following approximation: $\sin\theta = \tan\theta = \theta, \frac{dP_e}{dP_r} = \frac{P_e}{P_r}$. The expansion ratio is estimated as:

$$\frac{dP_e}{dP_r} = \frac{P_e}{P_r} = \frac{\theta_e}{\theta_r} = 1 - \frac{n_w}{n_a} \frac{d}{R} \frac{D}{d+D}. \quad (29)$$

Combine Eqs. (24) and (29), we have:

$$\lambda_{lower} \geq \frac{n_a}{n_w} \left| 1 - \frac{n_w}{n_a} \frac{d}{R} \frac{D}{d+D} \right|. \quad (30)$$

■

Parameter estimation The three parameters in Eq. (30), D , d and R , are estimated as following:

1. D is the distance from raindrop panel to background, $D > 1m$ is simply satisfied in outdoor vision system.

2. d is the distance from raindrop panel to the equivalent camera lens center. Usually, $d < 200mm$.

3. R is the radius curvature on the smoothest place of the raindrop. The bigger the raindrop is, the smoother the raindrop surface is. However a raindrop cannot be too big without sliding down a vertical panel. According to our observation (Fig. 1(a)), and static of size of falling raindrops in Garg. and Nayer's paper [1], usually, $R < 2.5mm$. Thus, $R < 5mm$ is a very safe estimation.

Substitute the estimation of D , d , R and $\frac{n_a}{n_w} = \frac{3}{4}$ to (20), we have:

$$\lambda_{lower} > 10. \quad (31)$$

The simplified version in the manuscript (Eq. (4)) is:

$$\mathcal{E}_\varphi > 10 \gg 1.$$

3. Proof of Areal Expansion Ratio

Lower bound of areal contraction ratio The lower bound of areal contraction ratio is:

$$\lambda_{lower}^2. \quad (32)$$

Mention that φ (Eq. (12)) is not conformal, the proof of Eq. (32) is given as following.

Proof Observing Eqs.(17) and (31), both eigenvalues of $(J_\varphi(u, v))^T J_\varphi(u, v)$ is greater than 0 which means $(J_\varphi(u, v))^T J_\varphi(u, v)$ is positive defined, and thus, $J_\varphi(u, v)$ is reversible at (u, v) :

$$\begin{pmatrix} \delta u \\ \delta v \end{pmatrix} = J_\varphi^{-1}(x, y) \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \quad (33)$$

And at the neighborhood of (u, v) , denoted as $V(u, v)$, φ is a one to one mapping which maps $V(u, v)$ to $V(x, y) = \varphi(V(u, v))$.

If we select a pixel in $V(x, y)$, its area can be defined as:

$$|\Delta \mathbf{x}| |\Delta \mathbf{y}| \sin \theta = |\Delta \mathbf{x}| |\Delta \mathbf{y}| \quad (34)$$

where $\Delta \mathbf{x} = (\Delta x, 0)^T$ and $\Delta \mathbf{y} = (0, \Delta y)^T$ denotes the directional width and height of the pixel, θ is the angle between $\Delta \mathbf{x}$ and $\Delta \mathbf{y}$. For a square pixel, $\Delta \mathbf{x}$ is orthogonal to $\Delta \mathbf{y}$, thus $\sin \theta = 1$.

For the given pixel area, its corresponding area on the raindrop can be calculated using Eq.(33):

$$|J_\varphi^{-1}(x, y) \Delta \mathbf{x}| |J_\varphi^{-1}(x, y) \Delta \mathbf{y}| \sin \theta' \quad (35)$$

where θ' is the angle between $J_\varphi^{-1}(x, y) \Delta \mathbf{x}$ and $J_\varphi^{-1}(x, y) \Delta \mathbf{y}$. Using the lower bound of linear expansion ratio in Eq.(31), we have:

$$\begin{aligned} |J_\varphi^{-1}(x, y) \Delta \mathbf{x}| |J_\varphi^{-1}(x, y) \Delta \mathbf{y}| \sin \theta' &\leq |J_\varphi(x, y)^{-1} \Delta \mathbf{x}| |J_\varphi(x, y)^{-1} \Delta \mathbf{y}| \\ &\leq |\lambda_{lower}^{-1} \Delta \mathbf{x}| |\lambda_{lower}^{-1} \Delta \mathbf{y}| = (\frac{1}{\lambda_{lower}})^2 |\Delta \mathbf{x}| |\Delta \mathbf{y}| \end{aligned} \quad (36)$$

Thus, the areal ratio is:

$$\lambda_{lower}^2. \quad (37)$$

■

The simplified version used in the manuscript (after Eq. (6)) is:

$$\mathcal{E}_\varphi^2 > 100 \gg 1.$$

References

- [1] K. Garg and S. Nayar. Photometric model of a raindrop. *CMU Technical Report*, 2003. 5
- [2] V. Zorich and R. Cooke. Mathematical analysis. *Springer*, 2004. 4